Solution to Problem Set 2

1. (a) According to the given graph of f, we have

$$f(-2) = 1, \qquad \lim_{x\to-2^-} f(x) = 3, \qquad \lim_{x\to-2^+} f(x) = 4,$$
$$f(2) = 4, \qquad \lim_{x\to2^-} f(x) = 2, \qquad \lim_{x\to2^+} f(x) = 0.$$

(b) (i) We have

$$g(0) = f(|0-2|) = f(2) = 4,$$
$$\lim_{x\to0^-} g(x) = \lim_{x\to0^-} f(|x-2|) = \lim_{t\to2^+} f(t) = 0,$$
$$\lim_{x\to0^+} g(x) = \lim_{x\to0^+} f(|x-2|) = \lim_{t\to2^-} f(t) = 2.$$

(ii) We have

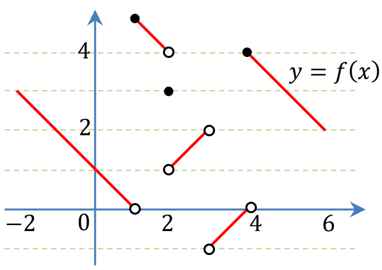
$$h(0) = f(|0|-2) = f(-2) = 1,$$
$$\lim_{x\to0^-} h(x) = \lim_{x\to0^-} f(|x|-2) = \lim_{t\to-2^+} f(t) = 4,$$
$$\lim_{x\to0^+} h(x) = \lim_{x\to0^+} f(|x|-2) = \lim_{t\to-2^+} f(t) = 4.$$

(iii) ⊙ Since the one-sided limits $\lim_{x\to0^-} f(x-2) = \lim_{t\to-2^-} f(t) = 3$ and $\lim_{x\to0^+} f(x-2) = \lim_{t\to-2^+} f(t) = 4$ are different, the limit $\lim_{x\to0} f(x-2)$ <u>does not exist</u>.

⊙ Since the one-sided limits $\lim_{x\to0^-} g(x) = 0$ and $\lim_{x\to0^+} g(x) = 2$ are different, the limit $\lim_{x\to0} g(x)$ <u>does not exist</u> either.

⊙ Since the one-sided limits $\lim_{x\to0^-} h(x) = 4$ and $\lim_{x\to0^+} h(x) = 4$ are the same, the limit $\lim_{x\to0} h(x)$ <u>exists</u> and $\lim_{x\to0} h(x) = 4$.

2. We consider the two one-sided limits separately.



⊙ As $x \to 2^-$, we have $4 - x \to 2^+$, so from the graph we see that $f(4-x) \to 1^+$.
Thus $3 - f(4-x) \to 2^-$, and so from the graph again, we have

$$\lim_{x\to2^-} f\big(3 - f(4-x)\big) = 4.$$

⊙ On the other hand, as $x \to 2^+$, we have $4 - x \to 2^-$, so from the graph we see that $f(4-x) \to 4^+$.
Thus $3 - f(4-x) \to -1^-$, and so from the graph again, we have

$$\lim_{x\to2^+} f\big(3 - f(4-x)\big) = 2.$$

Now $\lim_{x\to2^-} f\big(3 - f(4-x)\big) \neq \lim_{x\to2^+} f\big(3 - f(4-x)\big)$, so $\lim_{x\to2} f\big(3 - f(4-x)\big)$ does not exist.

3. Since $\lim\limits_{x\to 0}\frac{\ln(1+x)}{f(x)} = 1$, we have

$$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\ln(1+x)}{\frac{\ln(1+x)}{f(x)}}$$

When x approaches 0 but $x \neq 0$, we have $\ln(1+x) \neq 0$.

Limits in numerator and denominator both exist, and the denominator limit is non-zero.

$$= \frac{\lim\limits_{x\to 0}\ln(1+x)}{\lim\limits_{x\to 0}\frac{\ln(1+x)}{f(x)}} = \frac{\ln(1+0)}{1} = 0.$$

4. Since $\lim\limits_{x\to -1}\frac{x^3-ax^2-x+4}{x+1}$ exists as a real number, we write $\lim\limits_{x\to -1}\frac{x^3-ax^2-x+4}{x+1} = L$ where L is a real number. Then

$$\begin{aligned}\lim_{x\to -1}(x^3-ax^2-x+4) &= \lim_{x\to -1}\frac{x^3-ax^2-x+4}{x+1}\cdot(x+1)\\ &= \left(\lim_{x\to -1}\frac{x^3-ax^2-x+4}{x+1}\right)\left(\lim_{x\to -1}(x+1)\right)\\ &= (L)(-1+1) = 0.\end{aligned}$$

But on the other hand we also have

$$\lim_{x\to -1}(x^3-ax^2-x+4) = (-1)^3 - a(-1)^2 - (-1) + 4 = 4 - a.$$

These imply that $4 - a = 0$, so $a = 4$. Now to find the limit, we factorize the numerator to get

$$\begin{aligned}\lim_{x\to -1}\frac{x^3-4x^2-x+4}{x+1} &= \lim_{x\to -1}\frac{(x+1)(x-1)(x-4)}{x+1} = \lim_{x\to -1}(x-1)(x-4)\\ &= (-1-1)(-1-4) = 10.\end{aligned}$$

5. We first note that

$$x^2+x-2 = (x+2)(x-1)\begin{cases}\geq 0 & \text{if } x \in (-\infty,-2]\cup[1,+\infty)\\ < 0 & \text{if } x\in(-2,1)\end{cases}.$$

Thus

$$|x^2+x-2| = |(x+2)(x-1)| = \begin{cases}(x+2)(x-1) & \text{if } x \in (-\infty,-2]\cup[1,+\infty)\\ -(x+2)(x-1) & \text{if } x\in(-2,1)\end{cases}$$

and so

$$f(x) = \frac{|x^2+x-2|}{x+2} = \begin{cases}x-1 & \text{if } x \in (-\infty,-2)\cup[1,+\infty)\\ -(x-1) & \text{if } x\in(-2,1)\end{cases}.$$

Therefore we have

$$\begin{aligned}\lim_{x\to -2^-} f(x) &= \lim_{x\to -2^-}(x-1) = (-2)-1 = -3,\\ \lim_{x\to -2^+} f(x) &= \lim_{x\to -2^+} -(x-1) = -\big((-2)-1\big) = 3,\end{aligned}$$

and since $\lim\limits_{x\to -2^-} f(x) \neq \lim\limits_{x\to -2^+} f(x)$, the limit $\lim\limits_{x\to -2} f(x)$ does not exist.

6. We consider the following cases.

(i) If $a < -1$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a}(x^2+1) = a^2+1.$$

(ii) If $a = -1$, then since

$$\lim_{x\to -1^-} f(x) = \lim_{x\to -1^-}(x^2+1) = (-1)^2+1 = 2$$

and

$$\lim_{x\to -1^+} f(x) = \lim_{x\to -1^+} \sqrt{x+1} = \sqrt{(-1)+1} = 0$$

are not equal, $\lim_{x\to a} f(x) = \lim_{x\to -1} f(x)$ does not exist.

(iii) If $-1 < a < 1$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} \sqrt{x+1} = \sqrt{a+1}.$$

(iv) If $a = 1$, then since

$$\lim_{x\to 1^-} f(x) = \lim_{x\to 1^-} \sqrt{x+1} = \sqrt{1+1} = \sqrt{2}$$

and

$$\lim_{x\to 1^+} f(x) = \lim_{x\to 1^+} 2\sin\frac{\pi x}{4} = 2\sin\frac{\pi(1)}{4} = 2\cdot\frac{1}{\sqrt{2}} = \sqrt{2},$$

we have $\lim_{x\to a} f(x) = \lim_{x\to 1} f(x) = \sqrt{2}$.

(v) If $a > 1$, then

$$\lim_{x\to a} f(x) = \lim_{x\to a} 2\sin\frac{\pi x}{4} = 2\sin\frac{\pi a}{4}.$$

7. We have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}(x^3+a) = a$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\left(a\sin\frac{\pi x}{3}+b\right) = a\sin 0 + b = b$$

$$\lim_{x\to 2^-} f(x) = \lim_{x\to 2^-}\left(a\sin\frac{\pi x}{3}+b\right) = a\sin\frac{2\pi}{3}+b = \frac{\sqrt{3}}{2}a+b$$

$$\lim_{x\to 2^+} f(x) = \lim_{x\to 2^+} \log_2 x^{b+1} = \log_2 2^{b+1} = b+1.$$

Since $\lim_{x\to 0} f(x)$ and $\lim_{x\to 2} f(x)$ both exist, we must have

$$\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) \qquad \text{and} \qquad \lim_{x\to 2^-} f(x) = \lim_{x\to 2^+} f(x).$$

Therefore

$$a = b \qquad \text{and} \qquad \frac{\sqrt{3}}{2}a + b = b+1.$$

On solving these equations, we easily get

$$a = b = \frac{2}{\sqrt{3}}.$$

8. The coordinates of the point P are (x, x^2) in general, and the coordinates of the mid-point M of the line segment OP are $\left(\frac{x}{2}, \frac{x^2}{2}\right)$. Since the point Q always lies on the y-axis, its x-coordinate is always 0. Now let's denote the y-coordinate of Q by $k(x)$, which is a function of x. Since $QM \perp OP$, it follows that the product of the slopes of QM and OP equals -1, i.e.

$$\frac{k(x) - x^2/2}{0 - x/2} \cdot \frac{x^2 - 0}{x - 0} = -1.$$

This implies that

$$k(x) = \frac{1 + x^2}{2}.$$

As P approaches the origin, i.e. as $x \to 0$, we have

$$\lim_{x \to 0} k(x) = \lim_{x \to 0} \frac{1 + x^2}{2} = \frac{1 + 0^2}{2} = \frac{1}{2}.$$

This means that the y-coordinate of Q approaches $\frac{1}{2}$. Therefore as P approaches the origin, the point Q has a limiting position whose coordinates are $\left(0, \frac{1}{2}\right)$.

9. (a)

$$\begin{aligned}\lim_{x \to 1} \frac{x^6 - 1}{x - 1} &= \lim_{x \to 1} \frac{(x - 1)(x^5 + x^4 + x^3 + x^2 + x + 1)}{x - 1} \\ &= \lim_{x \to 1} (x^5 + x^4 + x^3 + x^2 + x + 1) \\ &= 1^5 + 1^4 + 1^3 + 1^2 + 1 + 1 = 6.\end{aligned}$$

(b)

$$\begin{aligned}\lim_{x \to -1} \frac{x^{1013} + 1}{x + 1} &= \lim_{x \to -1} \frac{(x + 1)(x^{1012} - x^{1011} + x^{1010} - \cdots + x^2 - x + 1)}{x + 1} \\ &= \lim_{x \to -1} (x^{1012} - x^{1011} + x^{1010} - \cdots + x^2 - x + 1) \\ &= (-1)^{1012} - (-1)^{1011} + (-1)^{1010} - \cdots + (-1)^2 - (-1) + 1 \\ &= 1013.\end{aligned}$$

(c) Let's use the change of variable $t = x^{\frac{1}{6}}$. Then $\sqrt[3]{x} = t^2$ and $\sqrt{x} = t^3$, and $t \to 1$ as $x \to 1$; so

$$\begin{aligned}\lim_{x \to 1} \frac{\sqrt[3]{x} - 1}{\sqrt{x} - 1} &= \lim_{t \to 1} \frac{t^2 - 1}{t^3 - 1} = \lim_{t \to 1} \frac{(t - 1)(t + 1)}{(t - 1)(t^2 + t + 1)} \\ &= \lim_{t \to 1} \frac{t + 1}{t^2 + t + 1} = \frac{1 + 1}{1^2 + 1 + 1} = \frac{2}{3}.\end{aligned}$$

(d)

$$\lim_{x\to 4}\frac{3x\sqrt{x+5}-12\sqrt{x+5}}{3-\sqrt{x+5}}=\lim_{x\to 4}\left[\frac{3(x-4)\sqrt{x+5}}{3-\sqrt{x+5}}\cdot\frac{3+\sqrt{x+5}}{3+\sqrt{x+5}}\right]=\lim_{x\to 4}\frac{9(x-4)\sqrt{x+5}+3(x-4)(x+5)}{3^2-(x+5)}$$

$$=\lim_{x\to 4}\frac{(x-4)\left[9\sqrt{x+5}+3(x+5)\right]}{4-x}=\lim_{x\to 4}\left[-9\sqrt{x+5}-3(x+5)\right]$$

$$=-9\sqrt{4+5}-3(4+5)$$

$$=-54.$$

(e)

$$\lim_{x\to 1}(x-1)\cot\pi x=\lim_{x\to 1}\frac{(x-1)\cos\pi x}{\sin\pi x}=\lim_{x\to 1}\frac{x-1}{\sin(\pi-\pi x)}\cdot\lim_{x\to 1}\cos\pi x$$

$$=\lim_{(1-x)\pi\to 0}\left[\frac{(1-x)\pi}{\sin(1-x)\pi}\cdot\frac{-1}{\pi}\right]\cdot\lim_{x\to 1}\cos\pi x$$

$$=\left(1\cdot\frac{-1}{\pi}\right)\cdot(-1)=\frac{1}{\pi}$$

(f) Since $-1\le\sin\frac{\pi}{x}\le 1$ for every $x\neq 0$, we have $e^{-1}\le e^{\sin\frac{\pi}{x}}\le e$ for every $x\neq 0$ and therefore

$$\sqrt{x^2+x^3}e^{-1}\le\sqrt{x^2+x^3}e^{\sin\frac{\pi}{x}}\le\sqrt{x^2+x^3}e$$

for every x near 0 and $x\neq 0$ (for every $x\in(-1,+\infty)\setminus\{0\}$, to be precise). Now since

$$\lim_{x\to 0}\sqrt{x^2+x^3}e^{-1}=\frac{1}{e}\lim_{x\to 0}|x|\sqrt{1+x}=\frac{1}{e}\cdot 0\cdot 1=0$$

and

$$\lim_{x\to 0}\sqrt{x^2+x^3}e=e\lim_{x\to 0}|x|\sqrt{1+x}=e\cdot 0\cdot 1=0,$$

by Squeeze Theorem we have

$$\lim_{x\to 0}\sqrt{x^2+x^3}e^{\sin\frac{\pi}{x}}=0.$$

(g) Since $-1\le\sin\frac{1}{x}\le 1$ for every $x\neq 0$, we have $-|x|\le x\sin\frac{1}{x}\le|x|$ for every $x\neq 0$. Now since

$$\lim_{x\to 0}-|x|=0\qquad\text{and}\qquad\lim_{x\to 0}|x|=0,$$

by Squeeze Theorem we have

$$\lim_{x\to 0}x\sin\frac{1}{x}=0.$$

Therefore,

$$\lim_{x\to 0}\frac{x^2\sin\frac{1}{x}}{\sin x}=\lim_{x\to 0}\left[\left(x\sin\frac{1}{x}\right)\cdot\left(\frac{x}{\sin x}\right)\right]=\left(\lim_{x\to 0}x\sin\frac{1}{x}\right)\left(\lim_{x\to 0}\frac{x}{\sin x}\right)=0\cdot 1=0.$$

(h) We first consider the limit $\lim_{x\to 0}\frac{\arctan 2x}{x}$ and use the change of variable $t = \arctan 2x$. Then $x = \frac{1}{2}\tan t$, and $t \to 0$ as $x \to 0$; so

$$\lim_{x\to 0}\frac{\arctan 2x}{x} = \lim_{t\to 0}\frac{t}{\frac{1}{2}\tan t} = \lim_{t\to 0}\left(\frac{t}{\sin t}\cdot 2\cos t\right) = \left(\lim_{t\to 0}\frac{t}{\sin t}\right)\left(\lim_{t\to 0}2\cos t\right) = 1\cdot 2\cos 0 = 2.$$

Therefore

$$\lim_{x\to 0}\frac{\arctan 2x}{\sin x} = \lim_{x\to 0}\left(\frac{\arctan 2x}{x}\cdot\frac{x}{\sin x}\right) = \left(\lim_{x\to 0}\frac{\arctan 2x}{x}\right)\left(\lim_{x\to 0}\frac{x}{\sin x}\right) = 2\cdot 1 = 2.$$

10. (a) Since $0 \le \sin^2 x \le 1$ for every $x > 0$, we have

$$0 \le \frac{\sin^2 x}{x^2} \le \frac{1}{x^2} \qquad \text{for every } x > 0.$$

Now $\lim_{x\to+\infty} 0 = 0$ and $\lim_{x\to+\infty}\frac{1}{x^2} = 0$, so by Squeeze Theorem we have

$$\lim_{x\to+\infty}\frac{\sin^2 x}{x^2} = 0.$$

Do not confuse $\lim_{x\to+\infty}\frac{\sin x}{x}$ with $\lim_{x\to 0}\frac{\sin x}{x}$. $\lim_{x\to+\infty}\frac{\sin x}{x}$ is not 1.

(b) Since $-1 \le \cos x \le 1$ for every $x > 1$, we have

$$\frac{x-1}{x-1} \le \frac{x+\cos x}{x-1} \le \frac{x+1}{x-1} \qquad \text{for every } x > 1.$$

Now

$$\lim_{x\to+\infty}\frac{x-1}{x-1} = \lim_{x\to+\infty} 1 = 1 \qquad \text{and} \qquad \lim_{x\to+\infty}\frac{x+1}{x-1} = \lim_{x\to+\infty}\frac{1+1/x}{1-1/x} = \frac{1+0}{1-0} = 1,$$

so by Squeeze Theorem we have

$$\lim_{x\to+\infty}\frac{x+\cos x}{x-1} = 1.$$

(c)

$$\lim_{x\to+\infty}\frac{\sqrt{x+\sqrt{x+\sqrt{x}}}}{\sqrt{x+1}} = \lim_{x\to+\infty}\frac{\sqrt{1+\frac{\sqrt{x+\sqrt{x}}}{x}}}{\sqrt{1+\frac{1}{x}}} = \lim_{x\to+\infty}\frac{\sqrt{1+\sqrt{\frac{1}{x}+\frac{1}{x^{3/2}}}}}{\sqrt{1+\frac{1}{x}}} = \frac{\sqrt{1+\sqrt{0+0}}}{\sqrt{1+0}} = 1$$

(d)

$$\lim_{x\to-\infty}\left(\sqrt{9x^2+3x+1}+3x\right) = \lim_{x\to-\infty}\left[\left(\sqrt{9x^2+3x+1}+3x\right)\cdot\frac{\sqrt{9x^2+3x+1}-3x}{\sqrt{9x^2+3x+1}-3x}\right]$$

$$= \lim_{x\to-\infty}\frac{(9x^2+3x+1)-(3x)^2}{\sqrt{9x^2+3x+1}-3x} = \lim_{x\to-\infty}\frac{3x+1}{-x\sqrt{9+3/x+1/x^2}-3x}$$

$$= \lim_{x\to-\infty}\frac{3+1/x}{-\sqrt{9+3/x+1/x^2}-3} = \frac{3+0}{-\sqrt{9+0+0}-3} = -\frac{1}{2}$$

(e) First note that with a change of variable $t = \frac{1}{x}$, we easily get $\lim\limits_{x\to+\infty} x\sin\frac{1}{x} = \lim\limits_{t\to 0^+} \frac{\sin t}{t} = 1$. So,

$$\lim_{x\to+\infty} \frac{2xe^x \sin\dfrac{1}{x} + e^x \cos\dfrac{1}{x}}{e^x + e^{-x}} = \lim_{x\to+\infty} \frac{2x\sin\dfrac{1}{x} + \cos\dfrac{1}{x}}{1 + e^{-2x}} = \frac{2\cdot \lim\limits_{x\to+\infty} x\sin\dfrac{1}{x} + \lim\limits_{x\to+\infty} \cos\dfrac{1}{x}}{1 + \lim\limits_{x\to+\infty} e^{-2x}}$$

$$= \frac{2\cdot 1 + \cos 0}{1+0} = 3.$$

(f) First note that

$$\lim_{x\to+\infty} (x - x^6) = \lim_{x\to+\infty} \underbrace{-x^6}_{\to-\infty} \underbrace{\left(1 - \frac{1}{x^5}\right)}_{\to 1} = -\infty.$$

Therefore with a change of variable $t = x - x^6$, we have

$$\lim_{x\to+\infty} \arctan(x - x^6) = \lim_{t\to-\infty} \arctan t = -\frac{\pi}{2}.$$

11. (a) After t minutes, the total volume of liquid in the tank is $(5000 + 25t)$ L, and the total amount of salt in the tank is $(25t)(30)$ g. So the concentration of salt in the tank is

$$C(t) = \frac{(25t)(30)}{5000 + 25t} = \frac{30t}{t + 200}$$

(in g/L).

(b) We have

$$\lim_{t\to+\infty} C(t) = \lim_{t\to+\infty} \frac{30t}{t+200} = \lim_{t\to+\infty} \frac{30}{1 + \dfrac{200}{t}} = \frac{30}{1+0} = 30.$$

Therefore as time passes, the concentration of salt in the tank will approach to a limit of 30 g/L.

12. (a) Note that since m and γ are both positive constants, we have $\lim\limits_{t\to+\infty} e^{-\frac{\gamma}{m}t} = 0$. Therefore,

$$\lim_{t\to+\infty} v(t) = \lim_{t\to+\infty} \left(ue^{-\frac{\gamma}{m}t} + \frac{mg}{\gamma}\left(1 - e^{-\frac{\gamma}{m}t}\right)\right)$$

$$= u\cdot 0 + \frac{mg}{\gamma}(1-0)$$

$$= \frac{mg}{\gamma},$$

which is a finite real number.

(b) The terminal velocity $\frac{mg}{\gamma}$ is independent of the initial velocity u of the object.

13. (a)

$$\lim_{x\to 1^-}\big(f(x)+g(x)\big)=\lim_{x\to 1^-}\left(\frac{2}{x^2-1}+\frac{1}{x-1}\right)=\lim_{x\to 1^-}\frac{2+(x+1)}{(x-1)(x+1)}=\lim_{x\to 1^-}\frac{\overbrace{x+3}^{\to 4}}{\underbrace{(x-1)}_{\to 0^-}\underbrace{(x+1)}_{\to 2}}=-\infty.$$

(b)

$$\lim_{x\to 1^+}\big(f(x)+g(x)\big)=\lim_{x\to 1^+}\left(\frac{2}{x^2-1}+\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{2+(x+1)}{(x-1)(x+1)}=\lim_{x\to 1^+}\frac{\overbrace{x+3}^{\to 4}}{\underbrace{(x-1)}_{\to 0^+}\underbrace{(x+1)}_{\to 2}}=+\infty.$$

(c)

$$\lim_{x\to 1^-}\big(f(x)-g(x)\big)=\lim_{x\to 1^-}\left(\frac{2}{x^2-1}-\frac{1}{x-1}\right)=\lim_{x\to 1^-}\frac{2-(x+1)}{(x-1)(x+1)}=\lim_{x\to 1^-}\frac{1-x}{(x-1)(x+1)}$$

$$=\lim_{x\to 1^-}\frac{-1}{x+1}=-\frac{1}{2}.$$

(d)

$$\lim_{x\to 1^+}\big(f(x)-g(x)\big)=\lim_{x\to 1^+}\left(\frac{2}{x^2-1}-\frac{1}{x-1}\right)=\lim_{x\to 1^+}\frac{2-(x+1)}{(x-1)(x+1)}=\lim_{x\to 1^+}\frac{1-x}{(x-1)(x+1)}$$

$$=\lim_{x\to 1^+}\frac{-1}{x+1}=-\frac{1}{2}.$$

14. (a)

$$\lim_{x\to 0}f(x)=\lim_{x\to 0}\left(\frac{f(x)}{x}\cdot x\right)=\left(\lim_{x\to 0}\frac{f(x)}{x}\right)\left(\lim_{x\to 0}x\right)=5\cdot 0=0.$$

(b) We have

$$\lim_{x\to 0^-}\frac{f(x)}{x^2}=\lim_{x\to 0^-}\frac{\overbrace{f(x)/x}^{\to 5}}{\underbrace{x}_{0^-}}=-\infty \qquad \text{but} \qquad \lim_{x\to 0^+}\frac{f(x)}{x^2}=\lim_{x\to 0^+}\frac{\overbrace{f(x)/x}^{\to 5}}{\underbrace{x}_{0^+}}=+\infty,$$

so $\lim_{x\to 0}\frac{f(x)}{x^2}$ does not exist.

(c) We have

$$\lim_{x\to 0^-}\frac{f(x)}{x^3}=\lim_{x\to 0^-}\frac{\overbrace{f(x)/x}^{\to 5}}{\underbrace{x^2}_{0^+}}=+\infty \qquad \text{and} \qquad \lim_{x\to 0^+}\frac{f(x)}{x^3}=\lim_{x\to 0^+}\frac{\overbrace{f(x)/x}^{\to 5}}{\underbrace{x^2}_{0^+}}=+\infty,$$

so $\lim_{x\to 0}\frac{f(x)}{x^3}=+\infty$.

15. (a) The domain of f is $\mathbb{R} \setminus \{1, -1\}$.

- Note that for every $x \in \mathbb{R} \setminus \{1, -1\}$ we have

$$f(x) = \frac{3x^5 + 3x^3}{2x^4 - 2} = \frac{3x^3(x^2+1)}{2(x-1)(x+1)(x^2+1)} = \frac{3x^3}{2(x-1)(x+1)},$$

from which we obtain

$$\lim_{x \to -1^-} f(x) = -\infty, \qquad \lim_{x \to -1^+} f(x) = +\infty, \qquad \lim_{x \to 1^-} f(x) = -\infty, \qquad \lim_{x \to 1^+} f(x) = +\infty.$$

Therefore the two lines $x = -1$ and $x = 1$ are both vertical asymptotes of the graph of f.

- Using long division, the function f can be rewritten as

$$f(x) = \frac{3x^5 + 3x^3}{2x^4 - 2} = \frac{3}{2}x + \frac{3x^3 + 3x}{2x^4 - 2}.$$

Thus we have

$$\lim_{x \to +\infty} \left(f(x) - \frac{3}{2}x \right) = \lim_{x \to +\infty} \frac{3x^3 + 3x}{2x^4 - 2} = \lim_{x \to +\infty} \frac{\frac{3}{x} + \frac{3}{x^3}}{2 - \frac{2}{x^4}} = \frac{0+0}{2-0} = 0$$

and similarly

$$\lim_{x \to -\infty} \left(f(x) - \frac{3}{2}x \right) = 0;$$

so the line $y = \frac{3}{2}x$ is a slant asymptote of the graph of f. The graph of f has no horizontal asymptotes.

(b) The domain of f is $\mathbb{R} \setminus \{3\}$.

- Since

$$\lim_{x \to 3^-} f(x) = \lim_{x \to 3^-} \frac{2x + e^x}{x - 3} = -\infty \qquad \text{and} \qquad \lim_{x \to 3^+} f(x) = \lim_{x \to 3^+} \frac{2x + e^x}{x - 3} = +\infty,$$

the line $x = 3$ is a vertical asymptote of the graph of f.

- Since

$$\lim_{x \to -\infty} f(x) = \lim_{x \to -\infty} \frac{2x + e^x}{x - 3} = \lim_{x \to -\infty} \frac{2 + \frac{e^x}{x}}{1 - \frac{3}{x}} = \frac{2+0}{1-0} = 2,$$

the line $y = 2$ is a horizontal asymptote of the graph of f. The graph of f has no slant asymptotes.

16. For every $x > 0$, we have

$$f(x) - mx = [f(x) - (mx + c)] + c$$

and

$$\frac{f(x)}{x} = \frac{f(x) - (mx + c)}{x} + \frac{mx + c}{x} = \frac{f(x) - (mx + c)}{x} + m + \frac{c}{x}.$$

Thus if $\lim_{x\to+\infty} [f(x) - (mx + c)] = 0$, then

$$\begin{aligned} \lim_{x\to+\infty} (f(x) - mx) &= \lim_{x\to+\infty} \{[f(x) - (mx + c)] + c\} \\ &= \lim_{x\to+\infty} [f(x) - (mx + c)] + \lim_{x\to+\infty} c \\ &= 0 + c = c \end{aligned}$$

and

$$\begin{aligned} \lim_{x\to+\infty} \frac{f(x)}{x} &= \lim_{x\to+\infty} \left(\frac{f(x) - (mx + c)}{x} + m + \frac{c}{x}\right) \\ &= \lim_{x\to+\infty} \frac{f(x) - (mx + c)}{x} + \lim_{x\to+\infty} \left(m + \frac{c}{x}\right) \\ &= 0 + (m + 0) = m. \end{aligned}$$